J Glob Optim (2008) 42:515–531
DOI 10.1007/s10898-007-9268-4

Sensitivity analysis for multivalued quasiequilibrium problems in metric spaces: Hölder continuity of solutions

Lam Quoc Anh · Phan Quoc Khanh

Received: 10 January 2007 / Accepted: 3 December 2007 / Published online: 25 December 2007

Abstract Hölder continuity and uniqueness of the solutions of general multivalued vector quasiequilibrium problems in metric spaces are established. The results are shown to be extensions of recent ones for equilibrium problems with some improvements. Applications in quasivariational inequalities, vector quasioptimization and traffic network problems are provided as examples for others in various optimization—related problems.

Keywords Multivalued vector quasiequilibrium problems · Hölder continuity · Solution uniqueness · Quasivariational inequalities · Traffic network problems · Quasioptimization

1 Introduction

The equilibrium problem, introduced by Blum and Oettli (1994) as a direct generalization of variational inequalities and optimization problems, has been proved to include many optimization—related problems. However, it does not contain quasivariational inequalities. The origin of the latter is the paper of Bensoussan et al. (1973) considering random impulse control problems and showing the need to deal with constraint sets depending on the state variables. A natural extension of the equilibrium problem to include quasivariational inequalities is the quasiequilibrium problem, which contains also various quasioptimization—related problems. Up to now there have been a great deal of works devoted to all aspects of quasiequilibrium problems like the solution existence, the sensitivity analysis and stability, solving methods, the solution uniqueness, etc. For the sensitivity analysis and stability we observe Bianchi and Pini (2003), Anh and Khanh (2004, 2007b, in press), which are devoted to semicontinuity

L. Q. Anh
Department of Mathematics, Teacher College, Cantho University, Cantho, Vietnam
e-mail: quocanh@ctu.edu.vn

P. Q. Khanh (✉)
Department of Mathematics, International University of Hochiminh City, Linh Trung, Thu Duc, Hochiminh City, Vietnam
e-mail: pqkhanh@hcmiu.edu.vn

of solution sets, and Ait Mansour and Riahi (2005), Anh and Khanh (2006, 2007a) which investigate the Hölder continuity of the unique solution of equilibrium problems.

The aim of the present paper is to extend Ait Mansour and Riahi (2005) and Anh and Khanh (2006, 2007a) to the case of general quasiequilibrium problems. When applying to quasivariational inequalities in reflexive Banach spaces our result sharpens that of Ait Mansour and Scrimali (online). Since the solution existence has been intensively studied, (see e.g., recent papers Giannessi 2000; Goh and Yang 1999; Hai and Khanh 2006, 2007a,b; Khaliq 2005 and references therein), we focalize our consideration on sensitivity analysis assuming always that the solutions exist in a neighborhood of the reference point.

The outline of the remainder of the paper is as follows. The rest of this section is devoted to explaining notions needed in the sequel. The main result is established in Sect. 2 followed by several direct sequences. In Sect. 3 we discuss applications of the main result.

Our notations are almost standard. We use $\|\cdot\|$ and $d(.,.)$ for the norm and metric in any normed space and metric space, respectively, (the context makes it clear what space is encountered). $d(x, A)$ is the distance from x to subset A in X. For a normed space X, X^* is the topological dual and $\langle .,. \rangle$ is the canonical pair. R_+ is the set of nonnegative real numbers. $B(x, r)$ denotes the closed ball of radius $r \geq 0$ and centered at x in a metric space X. int$C$ stands for the interior of a subset C.

Throughout the paper if not stated otherwise, let X, Z, Λ, M and N be metric spaces, Y be a metric linear space, $A \subseteq X$ be a nonempty subset and $C \subseteq Y$ have int $C \neq \emptyset$. Let $K : A \times \Lambda \to 2^X$ be a multifunction with nonempty values, $a : A \times N \to 2^Z$ and $F : X \times X \times Z \times M \to 2^Y$ be multifunctions. For subsets A and B under consideration we adopt the notations

$$\begin{aligned} &\mathrm{r}_1(A, B) \quad \text{means} \quad A \cap B \neq \emptyset;\\ &\mathrm{r}_2(A, B) \quad \text{means} \quad A \subseteq B;\\ &\varphi_1(A) = ((-A)\backslash l(A))^c;\\ &\varphi_2(A) = (-\mathrm{int}A)^c, \end{aligned}$$

where $l(A) = A \cap (-A)$ and $(.)^c$ is the complement of $(.)$. For each $\mathrm{r} \in \{\mathrm{r}_1, \mathrm{r}_2\}$, $\varphi \in \{\varphi_1, \varphi_2\}$, $\lambda \in \Lambda$, $\mu \in M$ and $\eta \in N$ consider the following quasiequilibrium problem:

($\mathrm{P_r}\varphi$) Find $\bar{x} \in K(\bar{x}, \lambda)$ and $\bar{x}^* \in a(\bar{x}, \eta)$ such that, for each $y \in K(\bar{x}, \lambda)$,

$$\mathrm{r}\left(F(\bar{x}, y, \bar{x}^*, \mu), \varphi(C)\right).$$

Let $S_{\mathrm{r}\varphi}(\lambda, \mu, \eta)$ be the solution set of ($\mathrm{P}_{\mathrm{r}\varphi}$) corresponding to λ, μ and η. Note that this problem statement is not quite explicit. However, it helps to unify the statements and proofs of assertions for four problems, ($\mathrm{P}_{\mathrm{r}\varphi}$) represents for each (λ, μ, η).

The following Hölder-related notions are in use in the sequel.

Definition 1.1 (i) (Classical) A multifunction $G : X \times \Lambda \to 2^X$ is said to be $(l_1.\alpha_1, l_2.\alpha_2)$ $-$Hölder at (x_0, λ_0) if there exist neighborhoods N of x_0 and U of λ_0 such that, $\forall x_1, x_2 \in N, \forall \lambda_1, \lambda_2 \in U$,

$$G(x_1, \lambda_1) \subseteq \left\{x \in X \mid \exists z \in G(x_2, \lambda_2), d(x, z) \leq l_1 d^{\alpha_1}(x_1, x_2) + l_2 d^{\alpha_2}(\lambda_1, \lambda_2)\right\}.$$

(ii) Let $G : X \times X \times \to 2^Y$ and $a : X \to 2^Z$ is called $h.\beta$-$\mathrm{r}_i.\varphi_2$-Hölder-strongly pseudomonotone relative to a in $S \subseteq X$, $i = 1, 2$, if $\forall x, y \in S : x \neq y$,

$$\begin{aligned} &[\exists x^* \in a(x), \mathrm{r}_i(G(x, y, x^*), \varphi_2(C))]\\ &\quad \Longrightarrow [\exists y^* \in a(y), G(y, x, y^*) + hB\left(0, d^{\beta}(x, y)\right) \subseteq -C], \end{aligned} \tag{1}$$

where $h \geq 0$ and $\beta > 0$. G is called $h.\beta$-$\mathrm{r}_i.\varphi_1$-Hölder-strongly pseudomonotone relative to a if (1) is replaced by

$$[\exists x^* \in a(x), \mathrm{r}_i(G(x, y, x^*), \varphi_1(C))] \\ \Longrightarrow [\exists y^* \in a(y), G(y, x, y^*) + hB\left(0, d^\beta(x, y)\right) \subseteq -C \backslash l(C)].$$

(iii) G is called $\mathrm{r}_i\varphi_k$- quasimonotone relative to a in $K \subseteq X$, for each $i = 1, 2$ and $k = 1, 2$, if $\forall x, y \in K : x \neq y$,

$$\forall x^* \in a(x), \bar{\mathrm{r}}_i\left(G(x, y, x^*), \varphi_k(C)\right) \Longrightarrow [\exists y^* \in a(y), \mathrm{r}_i\left(G(y, x, y^*), \varphi_k(C)\right)].$$

G is called $h.\beta$-$\mathrm{r}_1.\varphi_2$-Hölder-strongly monotone relative to a in $K \subseteq X$ if, $\forall x, y \in K$: $x \neq y$, $\exists x^* \in a(x)$, $\exists y^* \in a(y)$,

$$G(x, y, x^*) + G(y, x, y^*) + hd^\beta(x, y) \subseteq -C.$$

G is called $h.\beta$-$\mathrm{r}_2.\varphi_2$-Hölder-strongly monotone relative to a in $K \subseteq X$ if, $\forall x, y \in K$: $x \neq y$, $\forall x^* \in a(x)$, $\forall y^* \in a(y)$,

$$G(x, y, x^*) + G(y, x, y^*) + hd^\beta(x, y) \subseteq -C.$$

G is called $h.\beta$-$\mathrm{r}_1.\varphi_1$-Hölder-strongly monotone relative to a in $K \subseteq X$ if, $\forall x, y \in K$: $x \neq y$, $\exists x^* \in a(x)$, $\exists y^* \in a(y)$,

$$G(x, y, x^*) + G(y, x, y^*) + hd^\beta(x, y) \subseteq -C \backslash l(C).$$

G is called $h.\beta$-$\mathrm{r}_2.\varphi_1$-Hölder-strongly monotone relative to a in $K \subseteq X$ if, $\forall x$, $y \in K : x \neq y$, $\forall x^* \in a(x)$, $\forall y^* \in a(y)$,

$$G(x, y, x^*) + G(y, x, y^*) + hd^\beta(x, y) \subseteq -C \backslash l(C).$$

In special case where $X = Y = R$, $C = R_+$ and $f : R \times R \to R$, $a(x) \equiv \{x\}$, the above monotone properties reduce the following corresponding classical monotone properties.

Definition 1.2 Let $f : X \times X \to R$.

(i) f is called $h.\beta$-Hölder-strongly pseudomonotone in $K \subseteq X$ if, $\forall x, y \in K : x \neq y$,

$$[f(x, y) \geq 0] \Longrightarrow [f(y, x) + hB\left(0, d^\beta(x, y)\right) \leq 0],$$

where $h \geq 0$ and $\beta > 0$.

(ii) f is called quasimonotone in $K \subseteq X$ if, $\forall x, y \in K : x \neq y$,

$$[f(x, y) < 0] \Longrightarrow [f(y, x) \geq 0].$$

f is called $h.\beta$-Hölder-strongly monotone in $K \subseteq X$ if, $\forall x, y \in K : x \neq y$,

$$f(x, y) + f(y, x) + hd^\beta(x, y) \leq 0.$$

It is easy to see that if f is $h.\beta$-Hölder-strongly monotone in $K \subseteq X$, then f is $h.\beta$-Hölder-strongly pseudomonotone in $K \subseteq X$.

Example 1.1 Let $f: R \times R \to R$, $f(x, y) = y(x - y)$. Then it is not hard to that f is $\frac{1}{2}$.2-Hölder-strongly monotone in R and hence f is also $\frac{1}{2}$.2-Hölder-strongly pseudomonotone in R.

Example 1.2 Let $f: [0,1] \times [0,1] \to R$, $f(x,y) = \frac{y-x}{1+x}$. Then it is easy to that f is $\frac{1}{2}$.1-Hölder-strongly pseudomonotone in $[0,1]$. But f is not Hölder-strongly monotone in $[0,1]$, since $f(x,y) + f(y,x) = \frac{(y-x)^2}{(1+x)(1+y)} \geq 0$.

The following Hölder-related assumptions (cf. Anh and Khanh 2007a) will be essential for considering problem $(P_{r\varphi})$.

For the reference point $(\lambda_0, \mu_0, \eta_0) \in \Lambda \times M \times N$, there are neighborhoods $U(\lambda_0)$, $V(\mu_0)$ and $W(\eta_0)$ of λ_0, μ_0 and η_0, respectively, such that

(A1) $\forall \lambda \in U(\lambda_0), \forall \mu_1, \mu_2 \in V(\mu_0), \forall x, y \in E(\lambda) := \{x \in A \mid x \in K(x,\lambda)\}$: $x \neq y, \forall x_1^*, x_2^* \in a(E(\lambda), W(\eta_0))$,

$$F(x,y,x_1^*,\mu_1) \subseteq F(x,y,x_2^*,\mu_2) + B\left(0, d^{\theta}(x,y)\left(n_3 d^{\delta_3}(x_1^*,x_2^*) + n_4 d^{\delta_4}(\mu_1,\mu_2)\right)\right),$$

where $n_3, n_4, \delta_3, \delta_4$ and θ are nonnegative real numbers.

$(A2_{r_1\varphi})$ $\forall \mu \in V(\mu_0), \forall \eta \in W(\eta_0), \forall x, y \in E(U(\lambda_0)) : x \neq y$,

$$hd^{\beta}(x,y) \leq \inf_{x^* \in a(x,\eta)} \inf_{g \in F(x,y,x^*,\mu)} d(g, \varphi(C)) + \inf_{y^* \in a(y,\eta)} \inf_{f \in F(y,x,y^*,\mu)} d(f, \varphi(C)), \tag{2}$$

where $h > 0$, $\beta > \theta$.

$(A2_{r_2\varphi})$ is $(A2_{r_1\varphi})$ with (2) replaced by

$$hd^{\beta}(x,y) \leq \inf_{x^* \in a(x,\eta)} \sup_{g \in F(x,y,x^*,\mu)} d(g, \varphi(C)) + \inf_{y^* \in a(y,\eta)} \sup_{f \in F(y,x,y^*,\mu)} d(f, \varphi(C)).$$

Remark 1.1 These assumptions look seemingly complicated. But they are not hard to be checked as shown by examples below. We now make their meanings clearer.

(i) Assumption (A1) incorporates Hölder continuity with respect to state variables x, y and to parameter μ (in connection also with parameters λ and η). As explained in Anh and Khanh (2007a), this condition replaces particular orthogonality and linearity of variational inequalities in Hilbert spaces to ensure the Hölder continuity of the solution (see Theorem 2.1).

(ii) When $\varphi = \varphi_2$, assumptions $(A2_{r\varphi_2})$ become assumptions (A2a) and (A2b) in Anh and Khanh (2007a).

(iii) To explain Assumption $(A2_{r\varphi})$ we consider a single-valued real function (without parameters) $f: X \times X \to R$ for the sake of simplicity. Then the four assumptions $(A2_{r\varphi})$ collapse to the following assumption: $\forall x, y \in K \subseteq X : x \neq y$,

$$hd^{\beta}(x,y) \leq d(f(x,y), R_+) + d(f(y,x), R_+). \tag{3}$$

We have the following relation.

Proposition 1.1 (Anh and Khanh (2007a), Proposition 1.1).

(i) If $f: X \times X \to R$ satisfies (3) then f is $h.\beta$-Hölder-strongly pseudomonotone in K (the two types defined in Definition 1.1 (ii) *coincide in this case). Conversely, if f is $h.\beta$-Hölder-strongly pseudomonotone in K and quasimonotone in K, then f satisfies (3).*

(ii) If $f : X \times X \to R$ is $h.\beta$-Hölder-strongly monotone in $K \subseteq X$, then f satisfies (3).

Examples 1.1 and 1.2 in Anh and Khanh (2007a) interpret the lacking implications in Proposition 1.1.

In the general case we have the following relation.

Proposition 1.2 *(i) If F satisfies condition $(A2_{r\varphi})$ then $F(.,.,.,\mu)$ is $h.\beta$-r.φ-Hölder-strongly pseudomonotone relative to $a(.,\eta)$ in $E(U(\lambda_0))$ for each $\mu \in V(\mu_0)$ and $\eta \in W(\eta_0)$. Conversely, if $F(.,.,.,\mu)$ is $h.\beta-$r.φ-Hölder-strongly pseudomonotone relative to $a(.,\eta)$ in $E(U(\lambda_0))$ and quasimonotone relative to $a(.,\eta)$ in $E(U(\lambda_0))$ for each $\mu \in V(\mu_0)$ and $\eta \in W(\eta_0)$, then assumption $(A2_{r\varphi})$ is satisfied.*
(ii) If F is $h.\beta$-r.φ-Hölder-strongly monotone relative to $a(.,\eta)$ in $E(U(\lambda_0))$ for each $\mu \in V(\mu_0)$ and $\eta \in W(\eta_0)$, then assumption $(A2_{r\varphi})$ is fulfilled.

Proof Since $r \in \{r_1, r_2\}$ and $\varphi \in \{\varphi_1, \varphi_2\}$, we have in fact four cases corresponding to four different combinations of values of r and φ. However, the proof techniques are similar. We consider only the case where $r = r_1$ and $\varphi = \varphi_2$.

(i) If F satisfies assumption $(A2_{r\varphi})$ and $F(x, y, x^*, \mu) \cap (Y\backslash -\text{int}C) \neq \emptyset$, for some $x^* \in a(x,\eta)$, where $\mu \in V(\mu_0)$ and $\eta \in W(\eta_0)$. We show that $F(y, x, y^*, \mu) + hB(0, d^\beta(x, y)) \subseteq -C$, for some $y^* \in a(y,\eta)$. In this case, we see that $\inf_{x^* \in a(x,\eta)} \inf_{g\in F(x,y,x^*,\mu)} d(g, Y\backslash - \text{int}C) = 0$ and hence assumption $(A2_{r\varphi})$ yields that $\inf_{y^*\in a(y,\eta)} \inf_{f \in F(y,x,y^*,\mu)} d(f, Y\backslash - \text{int}C) \geq hd^\beta(x, y)$. Therefore, $F(y, x, y^*, \mu) + hB(0, d^\beta(x, y)) \subseteq -C$, for some $y^* \in a(y,\eta)$. Conversely, if F is $h.\beta$-$r_1.\varphi_2$-Hölder-strongly pseudomonotone and $r_1.\varphi_2$-quasimonotone in $E(U(\lambda_0))$. We show that assumption $(A2_{r_1\varphi_2})$ is fulfilled. Indeed, for $\mu \in V(\mu_0)$, $\eta \in W(\eta_0)$, if there is $x^* \in a(x,\eta)$ such that $F(x, y, x^*, \mu) \cap (Y\backslash -\text{int}C) \neq \emptyset$, since the $h.\beta$-$r_1.\varphi_2$-Hölder-strongly pseudomonotone of F we have $F(y, x, y^*, \mu) + hB(0, d^\beta(x, y)) \subseteq -C$, for some $y^* \in a(y,\eta)$ and hence

$$\inf_{y^*\in a(y,\eta)} \inf_{f\in F(y,x,y^*,\mu)} d(f, Y\backslash - \text{int}C) \geq hd^\beta(x, y),$$

i.e., assumption $(A2_{r_1\varphi_2})$ is fulfilled. If $\forall x^* \in a(x,\eta)$, $F(x, y, x^*, \mu) \cap (Y\backslash - \text{int}C) = \emptyset$, by the quasimonotone relative to $a(.,\eta)$ of $F(.,.,.,\mu)$ we see that $\exists y^* \in a(y,\eta)$ such that $F(y, x, y^*, \mu) \cap (Y\backslash - \text{int}C) \neq \emptyset$, the further arguments are the same as above.

(ii) F is $h.\beta$-$r_1.\varphi_2$-Hölder-strongly monotone relative to $a(.,\eta)$ in $E(U(\lambda_0))$ for each $\mu \in V(\mu_0)$ and $\eta \in W(\eta_0)$, there are $x^* \in a(x,\eta)$ and $y^* \in a(y,\eta)$ such that

$$F(x, y, x^*, \mu) + F(y, x, y^*, \mu) + hB(0, d^\beta(x, y)) \subseteq -C.$$

Therefore, for each $f \in F(y, x, y^*, \mu)$ and $g \in F(x, y, x^*, \mu)$ one has

$$d(g, Y\backslash - \text{int}C) + d(f, Y\backslash - \text{int}C) \geq d(g + f, Y\backslash - \text{int}C) \geq hd^\beta(x, y),$$

and hence assumption $(A2_{r_1\varphi_2})$ is fulfilled.

Examples 1.1 and 1.2 in Anh and Khanh (2007a) interpret the lacking implications in Proposition 1.2.

In this paper we use the following notation, for $C, D \subseteq X$,

$$\rho(C, D) = \sup_{x\in C, y\in D} d(x, y).$$

2 The main result

Theorem 2.1 *For problem* $(\mathrm{P}_{\mathrm{r}\varphi})$ *assume that solutions exist in a neighborhood of the considered point* $(\lambda_0, \mu_0, \eta_0) \in \Lambda \times M \times N$ *and the assumptions* (A1) *and* $(\mathrm{A2}_{\mathrm{r}\varphi})$ *are satisfied. Assume further that*

(i) $K(.,.)$ *is* $(l_1.\alpha_1, l_2.\alpha_2)-$*Hölder in* $E(U(\lambda_0)) \times \{\lambda_0\}$;
(ii) $\forall(\lambda, \mu, \eta) \in U(\lambda_0) \times V(\mu_0) \times W(\eta_0)$, $\forall x \in E(\lambda)$, $\forall x^* \in a(x, \eta)$, $F(x, ., x^*, \mu)$ *is* $n_2.\delta_2-$*Hölder in* $K(U(\lambda_0), \lambda)$;
(iii) $\forall x \in E(U(\lambda_0))$, $a(x, .)$ *is* $m.\gamma-$*Hölder at* η_0;
(iv) $\alpha_1\delta_2 = \beta$, $h > 2n_2 l_1^{\delta_2}$, $\beta > \theta$.

Then, the solution $x(\lambda, \mu, \eta)$ *of* $(\mathrm{P}_{\mathrm{r}\varphi})$ *is unique in a neighborhood of* $(\lambda_0, \mu_0, \eta_0)$ *and satisfies the following condition*

$$d(x(\lambda_1, \mu_1, \eta_1), x(\lambda_2, \mu_2, \eta_2)) \leq k_1 d^{\alpha_2\delta_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\delta_4/(\beta-\theta)}(\mu_1, \mu_2) + k_3 d^{\gamma\delta_3/(\beta-\theta)}(\eta_1, \eta_2),$$

where k_1, k_2 *and* k_3 *are positive constants depending on* $h, \beta, n_2, n_3, n_4, \theta, l_1, l_2, \ldots$

Proof Since $\mathrm{r} \in \{\mathrm{r}_1, \mathrm{r}_2\}$ and $\varphi \in \{\varphi_1, \varphi_2\}$, we have in fact four cases corresponding to four different combinations of values of r and φ. However, the proof techniques are similar. We consider only the case where $\mathrm{r} = \mathrm{r}_1$ and $\varphi = \varphi_2$. Let $\lambda_1, \lambda_2 \in U(\lambda_0)$, $\mu_1, \mu_2 \in V(\mu_0)$ and $\eta_1, \eta_2 \in W(\eta_0)$.

Step 1 We prove that, $\forall x(\lambda_1, \mu_1, \eta_1) \in S_{r_1\varphi_2}(\lambda_1, \mu_1, \eta_1)$, $\forall x(\lambda_1, \mu_2, \eta_1) \in S_{r_1\varphi_2}(\lambda_1, \mu_2, \eta_1)$,

$$d_1 := d(x(\lambda_1, \mu_1, \eta_1), x(\lambda_1, \mu_2, \eta_1)) \leq \left(\frac{n_4}{h - 2n_2 l_1^{\delta_2}}\right)^{1/(\beta-\theta)} d^{\delta_4/(\beta-\theta)}(\mu_1, \mu_2).$$

Let $x(\lambda_1, \mu_1, \eta_1) \neq x(\lambda_1, \mu_2, \eta_1)$ (if the equality holds then we are done). As $x(\lambda_1, \mu_1, \eta_1) \in K(x(\lambda_1, \mu_1, \eta_1), \lambda_1)$, $x(\lambda_1, \mu_2, \eta_1) \in K(x(\lambda_1, \mu_2, \eta_1), \lambda_1)$ and $K(.,.)$ is Hölder continuous, there are $x_1 \in K(x(\lambda_1, \mu_1, \eta_1), \lambda_1)$ and $x_2 \in K(x(\lambda_1, \mu_2, \eta_1), \lambda_1)$ such that

$$d(x(\lambda_1, \mu_1, \eta_1), x_2) \leq l_1 d^{\alpha_1}\left(x(\lambda_1, \mu_1, \eta_1), x(\lambda_1, \mu_2, \eta_1)\right), \tag{4}$$

$$d(x(\lambda_1, \mu_2, \eta_1), x_1) \leq l_1 d^{\alpha_1}\left(x(\lambda_1, \mu_1, \eta_1), x(\lambda_1, \mu_2, \eta_1)\right). \tag{5}$$

Since $x(\lambda_2, \mu_2, \eta_1)$ and $x(\lambda_2, \mu_2, \eta_1)$ are solutions of $(\mathrm{P}_{\mathrm{r}_1\varphi_2})$, there exist $x_1^* \in a(x(\lambda_1, \mu_1, \eta_1), \eta_1)$ and $x_2^* \in a(x(\lambda_1, \mu_2, \eta_1), \eta_1)$ such that

$$\exists z_1 \in F(x(\lambda_1, \mu_1, \eta_1), x_1, x_1^*, \mu_1) \cap (Y \backslash -\mathrm{int}C), \tag{6}$$

$$\exists z_2 \in F(x(\lambda_1, \mu_2, \eta_1), x_2, x_2^*, \mu_2) \cap (Y \backslash -\mathrm{int}C). \tag{7}$$

Assumption $(\mathrm{A2}_{\mathrm{r}_1\varphi_2})$ implies that

$$\inf_{x^* \in a(x(\lambda_1,\mu_1,\eta_1),\eta_1)} \inf_{g \in F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x^*,\mu_1)} d(g, Y \backslash -\mathrm{int}C) + \inf_{x^* \in a(x(\lambda_1,\mu_2,\eta_1),\eta_1)} \inf_{f \in F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x^*,\mu_1)} d(f, Y \backslash -\mathrm{int}C) \geq h d_1^{\beta}.$$

By (6) and (7), we have

$$\inf_{g\in F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_1^*,\mu_1)} d(g,z_1)+\inf_{f\in F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_1)} d(f,z_2)\geq hd_1^{\beta}.$$

Hence,

$$H\big(F(x(\lambda_1,\mu_1,\eta_1),x_1,x_1^*,\mu_1),F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_1^*,\mu_1)\big)$$
$$+H\big(F(x(\lambda_1,\mu_2,\eta_1),x_2,x_2^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_1)\big)\geq hd_1^{\beta},$$

where $H(.,.)$ is the Hausdorff distance. Consequently,

$$H\big(F(x(\lambda_1,\mu_1,\eta_1),x_1,x_1^*,\mu_1),F(x(\lambda_1,\mu_1,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_1^*,\mu_1)\big)$$
$$+H\big(F(x(\lambda_1,\mu_2,\eta_1),x_2,x_2^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_2)\big)$$
$$+H\big(F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_2),F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_1,\mu_1,\eta_1),x_2^*,\mu_1)\big)$$
$$\geq hd_1^{\beta}.$$

By assumption (A1) and (ii), one has

$$n_2d^{\delta_2}(x_1,x(\lambda_1,\mu_2,\eta_1))+n_2d^{\delta_2}(x_2,x(\lambda_1,\mu_1,\eta_1))+n_4d_1^{\theta}d^{\delta_4}(\mu_1,\mu_2)\geq hd_1^{\beta}.$$

Now (4) and (5) imply that

$$n_2l_1^{\delta_2}d_1^{\alpha_1\delta_2}+n_2l_1^{\delta_2}d_1^{\alpha_1\delta_2}+n_4d_1^{\theta}d^{\delta_4}(\mu_1,\mu_2)\geq hd_1^{\beta}.$$

Then assumption (iv) yields that

$$d_1^{\beta-\theta}\leq\left(\frac{n_4}{h-2n_2l_1^{\delta_2}}\right)d^{\delta_4}(\mu_1,\mu_2).$$

Setting $k_1=\left(\frac{n_4}{h-2n_2l_1^{\delta_2}}\right)^{\frac{1}{\beta-\theta}}$, we have

$$d_1\leq k_1d^{\frac{\delta_4}{\beta-\theta}}(\mu_1,\mu_2).$$

Step 2 Now we show that, $\forall x(\lambda_1,\mu_2,\eta_1)\in S_{r_1\varphi_2}(\lambda_1,\mu_2,\eta_1),\forall x(\lambda_2,\mu_2,\eta_1)\in S_{r_1\varphi_2}(\lambda_2,\mu_2,\eta_1)$,

$$d_2:=d(x(\lambda_1,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_1))\leq\left(\frac{2n_2l_2^{\delta_2}}{h-2n_2l_1^{\delta_2}}\right)^{1/\beta}d^{\alpha_2\delta_2/\beta}(\lambda_1,\lambda_2).$$

As before we can assume that $x(\lambda_1,\mu_2,\eta_1)\neq x(\lambda_2,\mu_2,\eta_1)$. Thanks to (i) we have $x_1'\in K(x(\lambda_2,\mu_2,\eta_1),\lambda_1)$ and $x_2'\in K(x(\lambda_1,\mu_2,\eta_1),\lambda_2)$ such that

$$d(x(\lambda_1,\mu_2,\eta_1),x_2')\leq l_2d^{\alpha_2}(\lambda_1,\lambda_2),\tag{8}$$

$$d(x(\lambda_2,\mu_2,\eta_1),x_1')\leq l_2d^{\alpha_2}(\lambda_1,\lambda_2).\tag{9}$$

By the Hölder continuity of $K(.,.)$ there are $x_1''\in K(x(\lambda_1,\mu_2,\eta_1),\lambda_1)$ and $x_2''\in K(x(\lambda_2,\mu_2,\eta_1),\lambda_2)$,

$$d(x_1',x_1'')\leq l_1d^{\alpha_1}\left(x(\lambda_1,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_1)\right),\tag{10}$$

$$d(x_2',x_2'')\leq l_1d^{\alpha_1}\left(x(\lambda_1,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_1)\right).\tag{11}$$

By the definition of $(\mathrm{P}_{\mathrm{r}_1\varphi_2})$, $x_1''^* \in a(x(\lambda_1,\mu_2,\eta_1),\eta_1)$ and $x_2''^* \in a(x(\lambda_2,\mu_2,\eta_1),\eta_1)$ exist such that one can find

$$z_1' \in F(x(\lambda_1,\mu_2,\eta_1), x_1'', x_1''^*, \mu_2) \cap (Y\backslash -\mathrm{int}C), \tag{12}$$

$$z_2' \in F(x(\lambda_2,\mu_2,\eta_1), x_2'', x_2''^*, \mu_2) \cap (Y\backslash -\mathrm{int}C). \tag{13}$$

It follows from assumption $(\mathrm{A2}_{\mathrm{r}_1\varphi_2})$ that

$$\inf_{x^*\in a(x(\lambda_1,\mu_2,\eta_1),\eta_1)} \inf_{g\in F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_1),x^*,\mu_2)} d(g, Y\backslash -\mathrm{int}C) + \inf_{x^*\in a(x(\lambda_2,\mu_2,\eta_1),\eta_1)} \inf_{f\in F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_1,\mu_2,\eta_1),x^*,\mu_2)} d(f, Y\backslash -\mathrm{int}C) \geq h d_2^{\beta}.$$

(12) and (13) then imply that

$$\inf_{g\in F(x(\lambda_1,\mu_2,\eta_1),x(\lambda_2,\mu_2,\eta_1),x_1''^*,\mu_2)} d(g, z_1') + \inf_{f\in F(x(\lambda_2,\mu_2,\eta_1),x(\lambda_1,\mu_2,\eta_1),x_2''^*,\mu_2)} d(f, z_2') \geq h d_2^{\beta}.$$

Consequently,

$$H\big(F(x(\lambda_1,\mu_2,\eta_1), x_1'', x_1''^*, \mu_2), F(x(\lambda_1,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_1), x_1''^*, \mu_2)\big) + H\big(F(x(\lambda_2,\mu_2,\eta_1), x_2'', x_2''^*, \mu_2), F(x(\lambda_2,\mu_2,\eta_1), x(\lambda_1,\mu_2,\eta_1), x_2''^*, \mu_2)\big) \geq h d_2^{\beta}.$$

and hence

$$\begin{aligned} &H\big(F(x(\lambda_1,\mu_2,\eta_1), x_1'', x_1''^*, \mu_2), F(x(\lambda_1,\mu_2,\eta_1), x_1', x_1''^*, \mu_2)\big)\\ &\quad + H\big(F(x(\lambda_1,\mu_2,\eta_1), x_1', x_1''^*, \mu_2), F(x(\lambda_1,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_1), x_1''^*, \mu_2)\big)\\ &\quad + H\big(F(x(\lambda_2,\mu_2,\eta_1), x_2'', x_2''^*, \mu_2), F(x(\lambda_2,\mu_2,\eta_1), x_2', x_2''^*, \mu_2)\big)\\ &\quad + H\big(F(x(\lambda_2,\mu_2,\eta_1), x_2', x_2''^*, \mu_2), F(x(\lambda_2,\mu_2,\eta_1), x(\lambda_1,\mu_2,\eta_1), x_2''^*, \mu_2)\big)\\ &\geq h d_2^{\beta}. \end{aligned}$$

The Hölder continuity of F assumed in (ii) implies that

$$\begin{aligned} &n_2 d^{\delta_2}(x_1'', x_1') + n_2 d^{\delta_2}(x_1', x(\lambda_2,\mu_2,\eta_1)) + n_2 d^{\delta_2}(x_2'', x_2')\\ &+ n_2 d^{\delta_2}(x_2', x(\lambda_1,\mu_2,\eta_1)) \geq h d_2^{\beta}. \end{aligned}$$

From (8), (9), (10) and (11) we have

$$n_2 l_1^{\delta_2} d_2^{\alpha_1\delta_2} + n_2 l_2^{\delta_2} d^{\alpha_2\delta_2}(\lambda_1,\lambda_2) + n_2 l_1^{\delta_2} d_2^{\alpha_1\delta_2} + n_2 l_2^{\delta_2} d^{\alpha_2\delta_2}(\lambda_1,\lambda_2) \geq h d_2^{\beta}.$$

It follows from assumption (iv) that

$$d_2^{\beta} \leq \left(\frac{2n_2 l_2^{\delta_2}}{h - 2n_2 l_1^{\delta_2}}\right) d^{\alpha_2\delta_2}(\lambda_1,\lambda_2).$$

Taking $k_2 = \left(\frac{2n_2 l_2^{\delta_2}}{h-2n_2 l_1^{\delta_2}}\right)^{\frac{1}{\beta}}$, one has

$$d_2 \leq k_2 d^{\frac{\alpha_2\delta_2}{\beta}}(\lambda_1,\lambda_2).$$

Step 3 We check the inequality, $\forall x(\lambda_2, \mu_2, \eta_1) \in S_{r_1\varphi_2}(\lambda_2, \mu_2, \eta_1)$, $\forall x(\lambda_2, \mu_2, \eta_2) \in S_{r_1\varphi_2}(\lambda_2, \mu_2, \eta_2)$,

$$d_3 := d(x(\lambda_2, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_2)) \le \left(\frac{n_3 m^{\delta_3}}{h - 2n_2 l_1^{\delta_2}} \right)^{\frac{1}{\beta-\theta}} d^{\frac{\gamma\delta_3}{\beta-\theta}}(\eta_1, \eta_2).$$

Assume that $x(\lambda_2, \mu_2, \eta_1) \neq x(\lambda_2, \mu_2, \eta_2)$. It follows from (i) the existence of $x_1''' \in K(x(\lambda_2, \mu_2, \eta_1), \lambda_2)$ and $x_2''' \in K(x(\lambda_2, \mu_2, \eta_2), \lambda_2)$ such that

$$d(x(\lambda_2, \mu_2, \eta_1), x_2''') \le l_1 d_3^{\alpha_1}, \tag{14}$$

$$d(x(\lambda_2, \mu_2, \eta_2), x_1''') \le l_1 d_3^{\alpha_1}. \tag{15}$$

Since $x(\lambda_2, \mu_2, \eta_1)$ and $x(\lambda_2, \mu_2, \eta_2)$ are solutions of $(\mathrm{P}_{r_1\varphi_2})$, there exist $x_1'''^* \in a(x(\lambda_2, \mu_2, \eta_1), \eta_1)$ and $x_2'''^* \in a(x(\lambda_2, \mu_2, \eta_2), \eta_2)$ such that we have

$$z_1'' \in F(x(\lambda_2, \mu_2, \eta_1), x_1''', x_1'''^*, \mu_2) \cap (Y \backslash -\mathrm{int}C), \tag{16}$$

$$z_2'' \in F(x(\lambda_2, \mu_2, \eta_2), x_2''', x_2'''^*, \mu_2) \cap (Y \backslash -\mathrm{int}C). \tag{17}$$

Assumption $(\mathrm{A2}_{r_1\varphi_2})$ implies that

$$\begin{aligned} &\inf_{x^* \in a(x(\lambda_2,\mu_2,\eta_1),\eta_1)} \inf_{g \in F(x(\lambda_2,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_2), x^*, \mu_2)} d(g, Y \backslash -\mathrm{int}C) \\ &+ \inf_{x^* \in a(x(\lambda_2,\mu_2,\eta_2),\eta_1)} \inf_{f \in F(x(\lambda_2,\mu_2,\eta_2), x(\lambda_2,\mu_2,\eta_1), x^*, \mu_2)} d(f, Y \backslash -\mathrm{int}C) \ge h d_3^{\beta}. \end{aligned} \tag{18}$$

Since $x_2'''^* \in a(x(\lambda_2, \mu_2, \eta_2), \eta_2)$, by (iii) there exists $x_1^* \in a(x(\lambda_2, \mu_2, \eta_2), \eta_1)$ such that

$$d(x_2'''^*, x_1^*) \le m d^{\gamma}(\eta_1, \eta_2). \tag{19}$$

It follows from (18) that

$$\begin{aligned} &\inf_{g \in F(x(\lambda_2,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_2), x_1'''^*, \mu_2)} d(g, Y \backslash -\mathrm{int}C) \\ &+ \inf_{f \in F(x(\lambda_2,\mu_2,\eta_2), x(\lambda_2,\mu_2,\eta_1), x_1^*, \mu_2)} d(f, Y \backslash -\mathrm{int}C) \ge h d_3^{\beta}. \end{aligned}$$

From (16) and (17) one has

$$\inf_{g \in F(x(\lambda_2,\mu_2,\eta_1), x(\lambda_2,\mu_2,\eta_2), x_1'''^*, \mu_2)} d(g, z_1'') + \inf_{f \in F(x(\lambda_2,\mu_2,\eta_2), x(\lambda_2,\mu_2,\eta_1), x_1^*, \mu_2)} d(f, z_2'') \ge h d_3^{\beta}.$$

Hence

$$\begin{aligned} &H\big(F(x(\lambda_2, \mu_2, \eta_1), x_1''', x_1'''^*, \mu_2), F(x(\lambda_2, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_2), x_1'''^*, \mu_2)\big) \\ &\quad + H\big(F(x(\lambda_2, \mu_2, \eta_2), x_2''', x_2'''^*, \mu_2), F(x(\lambda_2, \mu_2, \eta_2), x(\lambda_2, \mu_2, \eta_1), x_1^*, \mu_2)\big) \ge h d_3^{\beta}, \end{aligned}$$

and then

$$\begin{aligned} &H\big(F(x(\lambda_2, \mu_2, \eta_1), x_1''', x_1'''^*, \mu_2), F(x(\lambda_2, \mu_2, \eta_1), x(\lambda_2, \mu_2, \eta_2), x_1'''^*, \mu_2)\big) \\ &\quad + H\big(F(x(\lambda_2, \mu_2, \eta_2), x_2''', x_2'''^*, \mu_2), F(x(\lambda_2, \mu_2, \eta_2), x(\lambda_2, \mu_2, \eta_1), x_2'''^*, \mu_2)\big) \\ &\quad + H\big(F(x(\lambda_2, \mu_2, \eta_2), x(\lambda_2, \mu_2, \eta_1), x_2'''^*, \mu_2), F(x(\lambda_2, \mu_2, \eta_2), x(\lambda_2, \mu_2, \eta_1), x_1^*, \mu_2)\big) \\ &\ge h d_3^{\beta}. \end{aligned}$$

Assumptions (A1) and (ii) together imply that

$$n_2 d^{\delta_2}(x_1''', x(\lambda_2, \mu_2, \eta_2)) + n_2 d^{\delta_2}(x_2''', x(\lambda_2, \mu_2, \eta_1)) + n_3 d_3^{\theta} d^{\delta_3}(x_2'''^*, x_1^*) \geq h d_3^{\beta}.$$

From (14), (15) and (19), we obtain

$$n_2 l_1^{\delta_2} d_3^{\alpha_1 \delta_2} + n_2 l_1^{\delta_2} d_3^{\alpha_1 \delta_2} + n_3 m^{\delta_3} d_3^{\theta} d^{\gamma \delta_3}(\eta_1, \eta_2) \geq h d_3^{\beta}.$$

Assumption (iv) now yields that

$$d_3^{\beta-\theta} \leq \left(\frac{n_3 m^{\delta_3}}{h - 2 n_2 l_1^{\delta_2}} \right) d^{\gamma \delta_3}(\eta_1, \eta_2).$$

Setting $k_3 = \left(\frac{n_3 m^{\delta_3}}{h - 2 n_2 l_1^{\delta_2}} \right)^{\frac{1}{\beta-\theta}}$, we have

$$d_3 \leq k_3 d^{\frac{\gamma \delta_3}{\beta-\theta}}(\mu_1, \mu_2).$$

Step 4 Finally since, $\forall x(\lambda_1, \mu_1, \eta_1) \in S_{r_1 \varphi_2}(\lambda_1, \mu_1, \eta_1)$, $\forall x(\lambda_2, \mu_2, \eta_2) \in S_{r_1 \varphi_2}(\lambda_2, \mu_2, \eta_2)$

$$d(x(\lambda_1, \mu_1, \eta_1), x(\lambda_2, \mu_2, \eta_2)) \leq d_1 + d_2 + d_3,$$

we have

$$\rho \left(S_{r_1 \varphi_2}(\lambda_1, \mu_1, \eta_1), S_{r_1 \varphi_2}(\lambda_2, \mu_2, \eta_2) \right) \leq d_1 + d_2 + d_3.$$

Putting $(\lambda_1, \mu_1, \eta_1) = (\lambda_2, \mu_2, \eta_2)$ from this inequality one sees that $S_{r_1 \varphi_2}(\lambda_1, \mu_1, \eta_1)$ is a singleton. Similarly, $S_{r_1 \varphi_2}(\lambda_2, \mu_2, \eta_2)$ is also a singleton. Thus $(P_{r_1 \varphi_2})$ has a unique solution in a neighborhood of $(\lambda_0, \mu_0, \eta_0)$ and then the Hölder condition concluded in the theorem is obtained. □

Remark 2.1 In the case of equilibrium problems considered in Anh and Khanh (2007a), generalized monotonicity assumptions corresponding to $(A2_{r\varphi})$ ensure directly the solution uniqueness. However, for quasiequilibrium problems the above proof shows that this uniqueness is obtained by invoking all the assumptions together.

Examples 2.1 in Anh and Khanh (2007a) shows that assumptions $(A2_{r\varphi})$ are essential even in the special case where K depends only on λ.

Now we discuss some consequences of Theorem 2.1 for this special case, i.e. problems $(P_{r\varphi})$ becomes the corresponding equilibrium problem denoted by $(E_{r\varphi})$. When $r = r_1, \varphi = \varphi_2$ and $r = r_2, \varphi = \varphi_2$, Theorem 2.1 becomes Theorems 2.1 and 2.2, respectively, of Anh and Khanh (2007a) and sharpens Theorems 2.1 and 2.2, respectively, of Anh and Khanh (2006). To see this sharpening see Examples 2.4–2.6 in Anh and Khanh (2007a). Note that the cases where $r = r_1, \varphi = \varphi_1$ or $r = r_2, \varphi = \varphi_1$ are new even for the special case of $(E_{r\varphi})$. In addition, if $a(x, \eta) \equiv \{x\}$ and $F(x, y, x^*, \mu) = F(x, y, \mu)$ is single-valued, the special case of Theorem 2.1 improves Theorem 4.2 of Bianchi and Pini (2003) and Theorem 2.2.1 of Ait Mansour and Riahi (2005) (see also Examples 2.4–2.8 of Anh and Khanh (2007a) for detailed comparisons).

3 Applications

We will now apply the main result in Sect. 2 to some problems of importance. Since quasi-equilibrium problems include also many other problems, our result can clearly imply consequences for them.

3.1 Multivalued quasivariational inequalities

In this subsection, if not stated otherwise, let X be a reflexive Banach space, N and Λ be metric linear spaces, and $A \subseteq X$ be a nonempty subset. Let $K\colon A \times \Lambda \to 2^X$ and $a\colon A \times N \to 2^{X^*}$ be multifunctions with $K(x, \lambda)$ being closed and convex, $\forall(x, \lambda) \in A \times \Lambda$. For each $(\lambda, \eta) \in \Lambda \times N$ consider the quasivariational inequality problem
(QVI) Find $\bar{x} \in K(\bar{x}, \lambda)$ such that $\exists \bar{t} \in a(\bar{x}, \eta)$, $\forall y \in K(\bar{x}, \lambda)$,

$$\langle \bar{t}, y - \bar{x} \rangle \geq 0.$$

For each $(\lambda, \eta) \in \Lambda \times N$, by $S_{\mathrm{vi}}(\lambda, \eta)$ we denote the solution set of (QVI) at (λ, η).

To convert (QVI) to a special case of ($\mathrm{P}_{\mathrm{r}\varphi}$) set $Z = X^*$, $Y = R$, $C = R_+$ and $F(x, y, x^*) = \langle x^*, y - x \rangle$.

Corollary 3.1 *For* (QVI) *assume the solution existence in a neighborhood of* $(\lambda_0, \eta_0) \in \Lambda \times N$. *Assume further that there are neighborhoods* $U(\lambda_0)$ *of* λ_0, $W(\eta_0)$ *of* η_0 *such that we have* (i), (iii) *of Theorem 2.1 and*

(A2) $\forall \eta \in W(\eta_0), \forall x, y \in E(U(\lambda_0)) : x \neq y,$

$$h\|x - y\|^{\beta} \leq \inf_{g \in \langle a(x,\eta), y - x\rangle} d(g, R_+) + \inf_{f \in \langle a(y,\eta), x - y\rangle} d(f, R_+);$$

(a) *a is bounded in* $E(U(\lambda_0)) \times \{\eta_0\} : \|a(x, \eta)\| \leq n_2, \forall x \in E(U(\lambda_0)), \forall \eta \in W(\eta_0)$ *and* $E(U(\lambda_0))$ *is bounded*;
(b) $\alpha_1 = \beta, h > 2n_2 l_1$.

Then the solution $x(\lambda, \eta)$ *of* (QVI) *is unique in a neighborhood of* (λ_0, η_0) *and satisfies the Hölder condition*

$$\|x(\lambda_1, \eta_1) - x(\lambda_2, \eta_2)\| \leq k_1 d^{\alpha_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\gamma/\beta}(\mu_1, \mu_2).$$

Proof We simply check the assumptions of Theorem 2.1, except (i) and (iii). ($\mathrm{A2}_{\mathrm{r}\varphi}$) collapses to (A2) in this special case. For (A1) we see that

$$\begin{aligned} |F(x, y, x_1^*) - F(x, y, x_2^*)| &= |\langle x_1^*, y - x\rangle - \langle x_2^*, y - x\rangle| \\ &\leq \|y - x\|\|x_1^* - x_2^*\| \leq n_1\|x_1^* - x_2^*\|. \end{aligned}$$

Hence (A1) is fulfilled with $n_3 = n_1, \delta_3 = 1$ and $\theta = n_4 = \delta_4 = 0$. Since $\|a(x, \eta)\| \leq n_2$ in $E(U(\lambda_0)) \times W(\eta_0)$, assumption (ii) is satisfied with n_2 and $\delta_2 = 1$. Assumption (iv) becomes (b) in this case. □

Remark 3.1 Let $\bar{x} = x(\bar{\lambda}, \bar{\mu})$ be the solution of the variational inequality (VI) corresponding to (QVI), i.e. when K does not depend on x. Using similar arguments, Corollary 3.1 can be proved when replacing assumption (i) by the following Aubin property (known also as pseudo-Lipschitz property) of K around $(\bar{x}, \bar{\lambda})$ (but we have to add the maximal monotonicity of $a(., \eta)$): there exist neighborhoods P of $\bar{x}$, $\mathcal{V}(\bar{\lambda})$ of $\bar{\lambda}$ and $k > 0$ such that, $\forall \lambda_1, \lambda_2 \in \mathcal{V}(\bar{\lambda})$,

$$K(\lambda_1) \cap P \subseteq K(\lambda_2) + lB\,(0, d(\lambda_1, \lambda_2)).$$

Indeed, by classical arguments of existence theory for variational inequalities (cf. Anh and Khanh (2007a), we can consider that the solution $x(\lambda, \eta)$ to (VI) belongs to $K(\bar{\lambda}) \cap P$. Note that in this case $l_1 = 0$ and α_1 is arbitrary so we take $\alpha_1 = \beta$ and hence $\alpha_1.\delta_2 = \beta > 1$. Furthermore, in this case assumption (a) of Corollary 3.1 requires only a to be bounded in $E(U(\lambda_0)) \times \{\eta_0\}$ ($E(U(\lambda_0))$ needs not to be bounded) and hence assumption (A1) will be satisfied with $n_3 = \delta_3 = \theta = 1$ and $n_4 = \delta_4 = 0$. Namely we have the following consequence.

Corollary 3.2 *For* (VI) *assume the existence a neighborhood* $U(\lambda_0) \times W(\eta_0)$ *of* $(\lambda_0, \eta_0) \in \Lambda \times N$ *such that assumptions* (iii) *and* (A2) *of Corollary 3.1 are satisfied and assume further that*

(i′) *there is a neighborhood* P *of the solution* $x(\lambda_0, \eta_0)$ *such that,* $\forall \lambda, \lambda' \in U(\lambda_0)$,

$$K(\lambda) \cap P \subseteq K(\lambda') + lB\left(0, d^{\alpha}(\lambda, \lambda')\right)$$

(i.e. $K(.)$ *is* $l.\alpha-$*pseudo-Hölder at* λ_0*);*

(a′) a *is bounded in* $K(U(\lambda_0)) \times \{\eta_0\}$ *and* $\forall \eta \in W(\eta_0)$, $a(., \eta)$ *is maximal monotone*;

(b′) $\beta > 1$.

Then, in a neighborhood of (λ_0, η_0)*, the solution* $x(\lambda, \eta)$ *of* (VI) *is unique and satisfies the Hölder condition*

$$\|x(\lambda_1, \eta_1) - x(\lambda_2, \eta_2)\| \le k_1 d^{\alpha/\beta}(\lambda_1, \lambda_2) + k_2 d^{\xi/(\beta-1)}(\eta_1, \eta_2).$$

In the case where a is single—valued, Corollary 3.2 implies the following result

Corollary 3.3 *For* (VI) *assume that* a *is single—valued,* $x_0 := x(\lambda_0, \eta_0)$ *is a solution of* (VI) *at* (λ_0, η_0) *and that there is a neighborhood* $U(\lambda_0) \times W(\eta_0)$ *of* (λ_0, η_0) *such that*

(A2′) $a(., \eta)$ *is strongly monotone for each* $\eta \in W(\eta_0)$;

(i′) $K(.)$ *is pseudo-Lipschitz in* $U(\lambda_0)$;

(iii′) $a(., .)$ *is Lipschitz in* $P(x_0) \times W(\eta_0)$.

Then, in a neighborhood of (λ_0, η_0)*, the unique solution of* (VI) *satisfies the Hölder condition*

$$\|x(\lambda_1, \eta_1) - x(\lambda_2, \eta_2)\| \le k_1 d^{1/2}(\lambda_1, \lambda_2) + k_2 d(\eta_1, \eta_2).$$

Proof We check the assumptions of Corollary 3.1. (i′) holds with $\alpha = 1$. (A2) is satisfied with $\beta = 2$ by (A2′). (iii) is fulfilled with $\gamma = 1$ by (iii′). For (a′) we see that a is bounded since $a(., .)$ is Lipschitz continuous and $a(., \eta)$ is monotone; furthermore, since $a(., .)$ is single-valued and $a(., \eta)$ is monotone and demicontinuous, $a(., \eta)$ is maximal monotone by Lemma 2.13 of Kluge (1979). Finally, assumption (b′) is clearly satisfied. □

If X is a Hilbert space Corollary 3.3 collapses to Theorem 2.1 of Yen (1995).

Remark 3.2 As shown by Proposition 1.1, if a is single-valued, assumption (A2) is more relaxed than the $h.\beta$-Hölder-strong monotonicity of $a(., \eta)$ in $E(U(\lambda_0))$, $\forall \eta \in W(\lambda_0)$. When a is single-valued and $K(x, \lambda)$ is of a special linear form defined by the travel demands in a traffic network problem (see Subsect. 3.2), our problem (QVI) is reduced to the problem investigated in Ait Mansour and Scrimali (online). Theorem 2, the main result there, under assumptions similar to that of Corollary 3.1 with the mentioned monotonicity of $a(., \eta)$ replacing (A2), is weaker than Corollary 3.1 when $\beta = 2$.

3.2 A vector quasioptimization problem

Let X, Y, Λ, N, C and K be as for problem $(P_{r\varphi})$ in Sect. 1 and D: $X \times N \to 2^Y$ be a multifunction. For each $(\lambda, \eta) \in \Lambda \times N$, consider the following problem of (VOP_φ) finding $\bar{x} \in K(\bar{x}, \lambda)$ and $\bar{x}^* \in D(\bar{x}, \eta)$ such that, $\forall y \in K(\bar{x}, \lambda)$,

$$D(y, \eta) - \bar{x}^* \subseteq \varphi(C).$$

Recall that when K does not depend on x and $\varphi = \varphi_2$ such a point $\bar{x}$ is said to be a weak minimizer and $\bar{x}^*$ is a weak minimum of the vector optimization problem

$$\min D(y, \eta), s.t. \quad y \in K(\lambda),$$

and when K does not depend on x and $\varphi = \varphi_1$ such a point $\bar{x}$ is called an efficient minimizer and $\bar{x}^*$ is a Pareto minimum, i.e., there is no $y \in D(\bar{x}, \eta)$ such that

$$y - x^* \in (-C)\backslash l(C),$$

where $l(C) = C \cap (-C)$. Since the constraint set K here depends also on x, we have a quasioptimization problem.

To convert (VOP_φ) to a special case of $(\text{P}_{\text{r}_2\varphi})$ we simply set $Z = Y, M \equiv N$ and $F(x, y, x^*, \eta) = D(y, \eta) - x^*$. Then, from Theorem 2.1 we have (cf. also the proof of Theorem 2.1).

Corollary 3.4 *For* (VOP_φ) *assume that solutions exist in a neighborhood of* $(\lambda_0, \eta_0) \in \Lambda \times N$. *Assume further that there are neighborhoods* $U(\lambda_0)$ *of* λ_0 *and* $W(\eta_0)$ *of* η_0 *such that*

(A1) $\forall \lambda \in U(\lambda_0), \forall \eta_1, \eta_2 \in W(\eta_0), \forall y \in E(\lambda), \forall x_1^*, x_2^* \in D(E(\lambda), W(\eta_0))$,

$$D(y, \eta_1) - x_1^* \subseteq D(y, \eta_2) - x_2^* + \|y\|^\theta B\left(0, n_3\|x_1^* - x_2^*\| + n_4 d^{\delta_4}(\eta_1, \eta_2)\right),$$

where $n_3, n_4 > 0, \theta \geq 0$ *and* $\delta_4 > 0$;

(A2_φ) $\forall \eta \in W(\eta_0), \forall x, y \in E(U(\lambda_0)) : x \neq y$,

$$hd^\beta(x, y) \leq \inf_{x^* \in D(x,\eta)} \sup_{g \in D(y,\eta)-x^*} d(g, \varphi(C)) + \inf_{y^* \in D(y,\eta)} \sup_{f \in D(x,\eta)-y^*} d(f, \varphi(C)),$$

for $h > 0$ *and* $\beta > \theta$;

(i) $K(.,.)$ *is* $(l_1.\alpha_1, l_2.\alpha_2)$*−Hölder at* λ_0;
(ii) $\forall \eta \in W(\eta_0)$, $D(., \eta)$ *is* $n_2.\delta_2$*− Hölder in* $E(U(\lambda_0))$;
(iii) $\forall \lambda \in U(\lambda_0), \forall y \in E(\lambda)$, $D(y, .)$ *is* $m.\gamma$*− Hölder at* η_0.

Then, in a neighborhood of (λ_0, η_0), *the solution* $x(\lambda, \eta)$ *of* (VOP_φ), *is unique and satisfies the following condition*

$$d(x(\lambda_1, \eta_1), x(\lambda_2, \eta_2)) \leq k_1 d^{\alpha_2\delta_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\tau/(\beta-\theta)}(\eta_1, \eta_2),$$

where $\tau := \min\{\delta_4, \gamma\}$, k_1 *and* k_2 *are positive constants depending on* $h, \beta, m, \theta, \ldots$

3.3 Applications to traffic network problems

Wardrop (1952) introduced a notion of equilibrium flows for transportation network problems and proved basic traffic network principles. Until now many contributions have developed this research direction in various aspects. We would notice some points in the development process. Smith (1979) began the variational approach by proving that the Wardrop equilibria are just solutions of variational inequalities corresponding to the traffic network problems. In De Luca (1995) and Maugeri (1995) the travel demands of the problem was proposed to depend on the equilibrium vector flows to meet the variety of practical situations. These elastic demands led to the fact that the traffic problem corresponded to a quasivariational (not variational) inequality. De Luca (1995) and Maugeri (1995) considered the travel costs being multifunctions of the path flows. Khanh and Luu (2004, 2005) extended the notion of Wardrop's traffic equilibria to this case. Efforts have been devoted mainly to the solution existence. Recently, in Ait Mansour and Scrimali (online), the stability of the problem in terms of the Hölder continuity of the solution with respect to the perturbing parameters is studied. In this subsection we apply the results in Sect. 2 to establish sharpened Hölder continuity results for more general traffic network problems with multivalued costs.

We describe first our traffic problem. Let the network consist of nodes and links (or arcs). Let $W = (W_1, \ldots, W_l)$ be the set of pairs, each of them consists of an origin node and a destination node, (O/D pairs for short). Assume that P_j, $j = 1, \ldots, l$, is the set of paths connecting the pair W_j and that P_j contains $r_j \geq 1$ paths. Let $m = r_1 + \cdots + r_l$ and $f = (f_1, \ldots, f_m)$ denote the path vector flow. Giannessi (1980) proposed that restrictions of the capacity of the paths must be considered. Hence we assume that the constraint of the capacity of paths is of the form

$$A = \{f \in R^m : \gamma_s \leq f_s \leq \Gamma_s, s = 1, \ldots, m\},$$

where γ_s and Γ_s are given nonnegative numbers. Let the cost vector $T(f, \mu) = (T_1(f, \mu), \ldots, T_m(f, \mu))$ be a multifunction of flow f and perturbing parameter μ. The generalization of the Wardrop equilibrium for the multivalued cost case is as follows.

Definition 3.1 (i) A path vector flow f is said to be a weak equilibrium vector flow if $\forall W_j, \forall q \in P_j, \forall s \in P_j, \exists t \in T(f, \mu)$,

$$t_q < t_s \Longrightarrow f_q = \Gamma_q \quad \text{or} \quad f_s = \gamma_s,$$

where $j = 1, \ldots, l$ and $q, s \in \{1, \ldots, m\}$ are among r_j paths corresponding to P_j.

(ii) A path vector flow f is called a strong equilibrium vector flow if (i) is satisfied with $\exists t \in T(f, \mu)$ being replaced by $\forall t \in T(f, \mu)$.

Assume further that the travel demand g_j of the O/D pair W_j depends on the equilibrium vector flow h as explained in De Luca (1995) and Maugeri (1995) and also on a perturbing parameter $\lambda \in \Lambda$: $g_j(h, \lambda)$. Denote the travel vector demand by $g = (g_1, \ldots, g_l)$ and use the Kronecker numbers

$$\phi_{js} = \begin{cases} 1, & \text{if } s \in P_j, \\ 0, & \text{if } s \notin P_j, \end{cases}$$
$$\phi = \{\phi_{js}\}, j = 1, \ldots, l; s = 1, \ldots, m.$$

Then the set of all feasible path vector flows is

$$K(h, \lambda) = \{f \in A \mid \phi z = g(h, \lambda)\},$$

where ϕ is called the O/D pair–path incidence matrix.

Note that the traffic problem formulated in terms of path flow variables as above needs not the additivity of the travel cost, i.e. a path cost may not be equal to the sum of the link costs for all links involved in the path. For formulations based on link variables such additivity must be assumed.

Observe that a feasible path flow vector $\bar{f}$ is a weak (or strong) equilibrium flow vector if $\bar{f}$ is a solution of the following quasivariational inequality, respectively, (see Khanh and Luu 2004):

(TNP_{r_1}) Find $\bar{f} \in K(\bar{f}, \lambda)$ such that, $\forall f \in K(\bar{f}, \lambda)$, $\exists \bar{t} \in T(\bar{f}, \mu)$,

$$\langle \bar{t}, f - \bar{f} \rangle \geq 0.$$

(TNP_{r_2}) Find $\bar{f} \in K(\bar{f}, \lambda)$ such that $\forall f \in K(\bar{f}, \lambda)$, $\forall \bar{t} \in T(\bar{f}, \mu)$ such that

$$\langle \bar{t}, f - \bar{f} \rangle \geq 0.$$

Lemma 3.5 (Proposition 1, Ait Mansour and Scrimali (online)) *If $g(.,.)$ is $(L_1.\alpha_1, L_2.\alpha_2)$-Hölder at (x_0, λ_0) then there are l_1, l_2 such that K is $(l_1.\alpha_1, l_2.\alpha_2)$-Hölder at (x_0, λ_0).*

Setting $X = Z = R_m, N \equiv \Lambda, Y = R, C = R_+, a(x, \lambda) = Z$ and $F(h, f, h^*, \mu) = \langle T(h, \mu), f - h \rangle$. Then our problems ($P_{r\varphi_1}$) coincides with ($P_{r\alpha_2}$); ($P_{r_1\varphi}$) and ($P_{r_2\varphi}$) becomes ($TNP_{r_1}$) and ($TNP_{r_2}$), respectively. Hence we can derive the Hölder continuity of (TNP_{r_1}) and (TNP_{r_2}) from Theorem 2.1 as follows.

First note that, since in this case $F(h, f, h^*, \mu)$ does not depend on h^* and the problem is scalar (hence $\varphi_1 = \varphi_2$), assumptions ($A_{r_1\varphi}$) and ($A_{r_2\varphi}$) collapse to the following assumptions, respectively,

(A_{r_1}) $\forall \mu \in V(\mu_0), \forall x, y \in E(U(\lambda_0)) : x \neq y$,

$$h\|x - y\|^\beta \leq \inf_{g \in \langle T(x,\mu), y-x \rangle} d(g, R_+) + \inf_{f \in \langle T(y,\mu), x-y \rangle} d(f, R_+).$$

(A_{r_2}) $\forall \mu \in V(\mu_0), \forall x, y \in E(U(\lambda_0)) : x \neq y$,

$$h\|x - y\|^\beta \leq \sup_{g \in \langle T(x,\mu), y-x \rangle} d(g, R_+) + \sup_{f \in \langle T(y,\mu), x-y \rangle} d(f, R_+).$$

Corollary 3.6 *For* (TNP_r) *assume that there are neighborhoods $U(\lambda_0)$ of λ_0 and $V(\mu_0)$ of μ_0 such that assumption* (A_r) *is satisfied. Assume further that*

(a) *$\forall f \in E(U(\lambda_0))$, $T(f, .)$ is $n.\delta$-Hölder at μ_0 and $T(.,.)$ is bounded: $\forall f \in E(U(\lambda_0))$, $\forall \mu \in V(\mu_0), \forall t \in T(f, \mu), \|t\| \leq M$; and $E(U(\lambda_0))$ is bounded: $\forall f \in E(U(\lambda_0))$, $\|f\| \leq N$;*
(b) *g is $(L_1.\alpha_1, L_2.\alpha_2)$-Hölder in $E(U(\lambda_0)) \times \{\lambda_0\}$;*
(c) *$\alpha_1 = \beta$ and $h > 2n_2l_1$.*

Then, in a neighborhood of (λ_0, μ_0), the solution of (TNP_r) *is unique and satisfies the following Hölder condition*

$$d(f(\lambda_1, \mu_1), f(\lambda_2, \mu_2)) \leq k_1 d^{\alpha_2/\beta}(\lambda_1, \lambda_2) + k_2 d^{\delta/\beta}(\mu_1, \mu_2),$$

where k_1 and k_2 are positive constants depending on h, β, n, δ, etc.

Proof Taking into account Lemma 3.5, it suffices to check only assumptions (ii) and (A1) of Theorem 2.1 (assumption (iii) is satisfied with any $m \geq 0, \gamma \geq 0$). For any $F_1 = \langle t, f_1 - h \rangle$ we take $F_2 = \langle t, f_2 - h \rangle$. Then

$$\|F_1 - F_2\| \leq \langle t, f_1 - f_2 \rangle \leq M\|f_1 - f_2\|.$$

Hence

$$\langle T(h, \mu), f_1 - h \rangle \subseteq \langle T(h, \mu), f_2 - h \rangle + M\|f_1 - f_2\|B(0, 1),$$

i.e. assumption (ii) is fulfilled with $n_2 = M$ and $\delta_2 = 1$. Similarly, it is not hard to see that (A1) is satisfied with $\theta = 0, n_4 = nN, \delta_4 = \delta, n_3 = 0$ and δ_3 is any nonnegative numbers. □

Remark 3.3 In Ait Mansour and Scrimali (online) the special case of our traffic network problem, where T is single-valued, is investigated. Instead of (A_r) a strong monotonicity of T (which is stricter than (A_r)) is assumed.

Acknowledgements This work was supported in part by the National Basic Research Program in Natural Sciences of Ministry of Science and Technology of Vietnam. The authors would like to thank the referees for their remarks and suggestions, which helped to improve the paper.

References

Ait Mansour, M., Scrimali, L.: Hölder continuity of solutions to elastic traffic network models. J. Glob. Optim., online.
Ait Mansour, M., Riahi, H.: Sensitivity analysis for abstract equilibrium problems. J. Math. Anal. Appl. **306**, 684–691 (2005)
Anh, L.Q., Khanh, P.Q.: Semicontinuity of the solution set of parametric multivalued vector quasiequilibrium problems. J. Math. Anal. Appl. **294**, 699–711 (2004)
Anh, L.Q., Khanh, P.Q.: On the Hölder continuity of solutions to parametric multivalued vector equilibrium problems. J. Math. Anal. Appl. **321**, 308–315 (2006)
Anh, L.Q., Khanh, P.Q.: Uniqueness and Hölder continuity of the solution to multivalued equilibrium problems in metric spaces. J. Glob. Optim. **37**, 449–465 (2007a)
Anh, L.Q., Khanh, P.Q.: On the stability of the solution sets of general multivalued vector quasiequilibrium problems. J. Optim. Theory Appl. **135**, 271–284 (2007b)
Anh, L.Q., Khanh, P.Q.: Various kinds of semicontinuity and the solution sets of parametric multivalued symmetric vector quasiequilibrium problems. J. Glob. Optim. (in press)
Bensoussan, A., Goursat, M., Lions, J.L.: Contrôle impulsionnel et inéquations quasivariationnelle. C. R. Acad. Sci. Paris, Sér. A **276**, 1279–1284 (1973)
Bianchi, M., Pini, R.: A note on stability for parametric equilibrium problems. Oper. Res. Lett. **31**, 445–450 (2003)
Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
De Luca, M.: Generalized quasivariational inequalities and traffic equilibrium problems. In: Giannessi, F., Maugeri, A. (eds.) Variational Inequalities and Network Equilibrium Problems, pp. 45–54. Plenum Press, New York (1995)
Giannessi, F.: Theorems of the alternative, quadratic programs, and complementarity problems. In: Cottle, R.W., Giannessi, F., Lions, J.L. (eds.) Variational Inequalities and Complementarity Problems, pp. 151–186. John Wiley and Sons, New York (1980)
Giannessi, F.: Vector Variational Inequalities and Vector Equilibria. Kluwer, Dordrecht (2000)
Goh, C.J., Yang, X.Q.: Vector equilibrium problem and vector optimization. European J. Oper. Res. **116**, 615–628 (1999)
Hai, N.X., Khanh, P.Q.: Systems of multivalued quasiequilibrium problems. Adv. Nonlinear Variat. Inequal. **9**, 97–108 (2006)
Hai, N.X., Khanh, P.Q.: The solution existence of general variational inclusion problems. J. Math. Anal. Appl. **328**, 1268–1277 (2007a)
Hai, N.X., Khanh, P.Q.: Existence of solutions to general quasiequilibrium problems and applications. J. Optim. Theory Appl. **133**, 317–327 (2007b)
Khaliq, A.: Implicit vector quasiequilibrium problems with applications to variational inequalities. Nonlinear Anal. **63**, 1823–1831 (2005)
Khanh, P.Q., Luu, L.M.: On the existence of solutions to vector quasivariational inequalities and quasicomplementarity problems with applications to traffic network equilibria. J. Optim. Theory Appl. **123**, 533–548 (2004)
Khanh, P.Q., Luu, L.M.: Some existence results for vector quasivariational inequalities involving multifunctions and applications to traffic equilibrium problems. J. Glob. Optim. **32**, 551–568 (2005)
Kluge, R.: Nichtlineare Variationsungleichungen und Extremalaufgaben. Wissenschaften, Berlin (1979)

Maugeri, A.: Variational and quasivariational inequalities in network flow models: recent developments in theory and algorithms. In: Giannessi, F., Maugeri, A. (eds.) Variational Inequalities and Network Equilibrium Problems, pp. 195–211. Plenum Press, New York (1995)
Smith, M.J.: The existence, uniqueness and stability of traffic equilibrium. Transport. Res. **138**, 295–304 (1979)
Wardrop, J.G.: Some theoretical aspects of road traffic research. Proc. Inst. Civil Eng. Part II, 325–378 (1952)
Yen, N.D.: Hölder continuity of solutions to parametric variational inequalities. Appl. Math. Optim. **31**, 245–255 (1995)